Direct CT-Scan to Finite Element Modeling Using a 3D Fictitious Domain Method with an Application to Biomechanics

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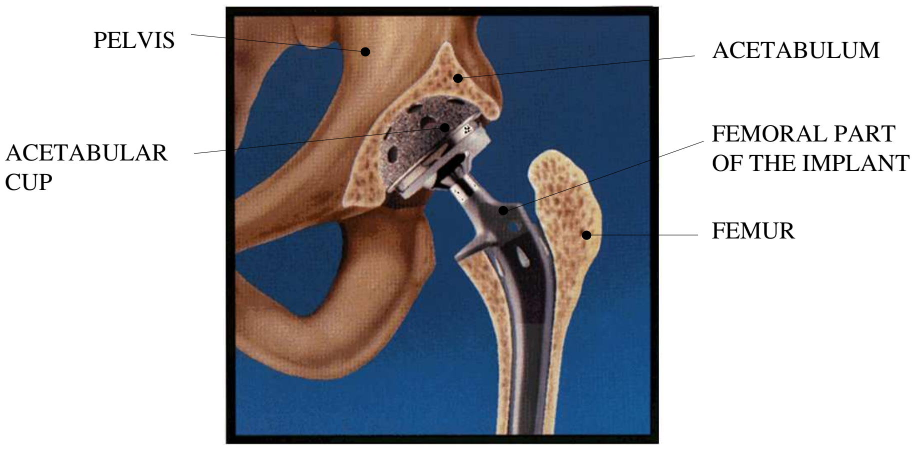
Driving Model Problem - THR Surgery

THR: Total Hip Replacement

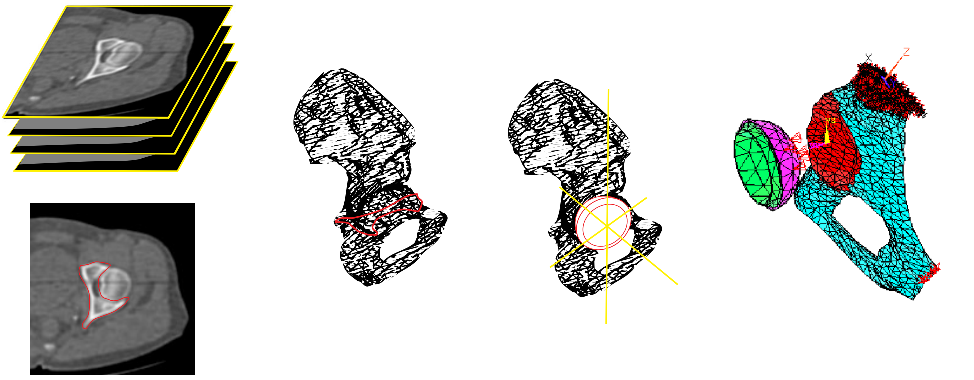
- 200,000 THR/year in the US
- 800,000 THR/year worldwide
- Problems: fracture, dislocation incidence, loss of contact, etc

TOTAL HIP REPLACEMENT - (THR)

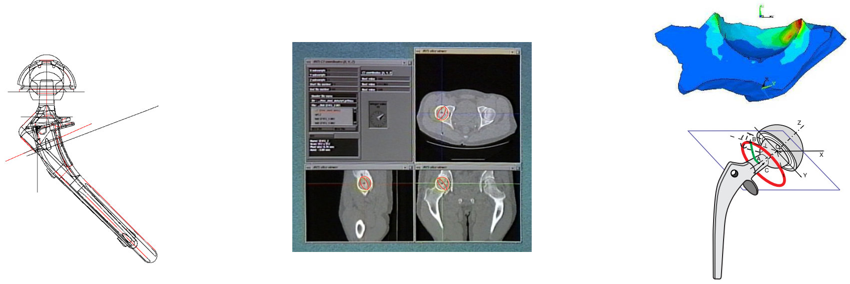


COMPUTER-ASSISTED THR

Modeling - Image Processing & Computational Geometry

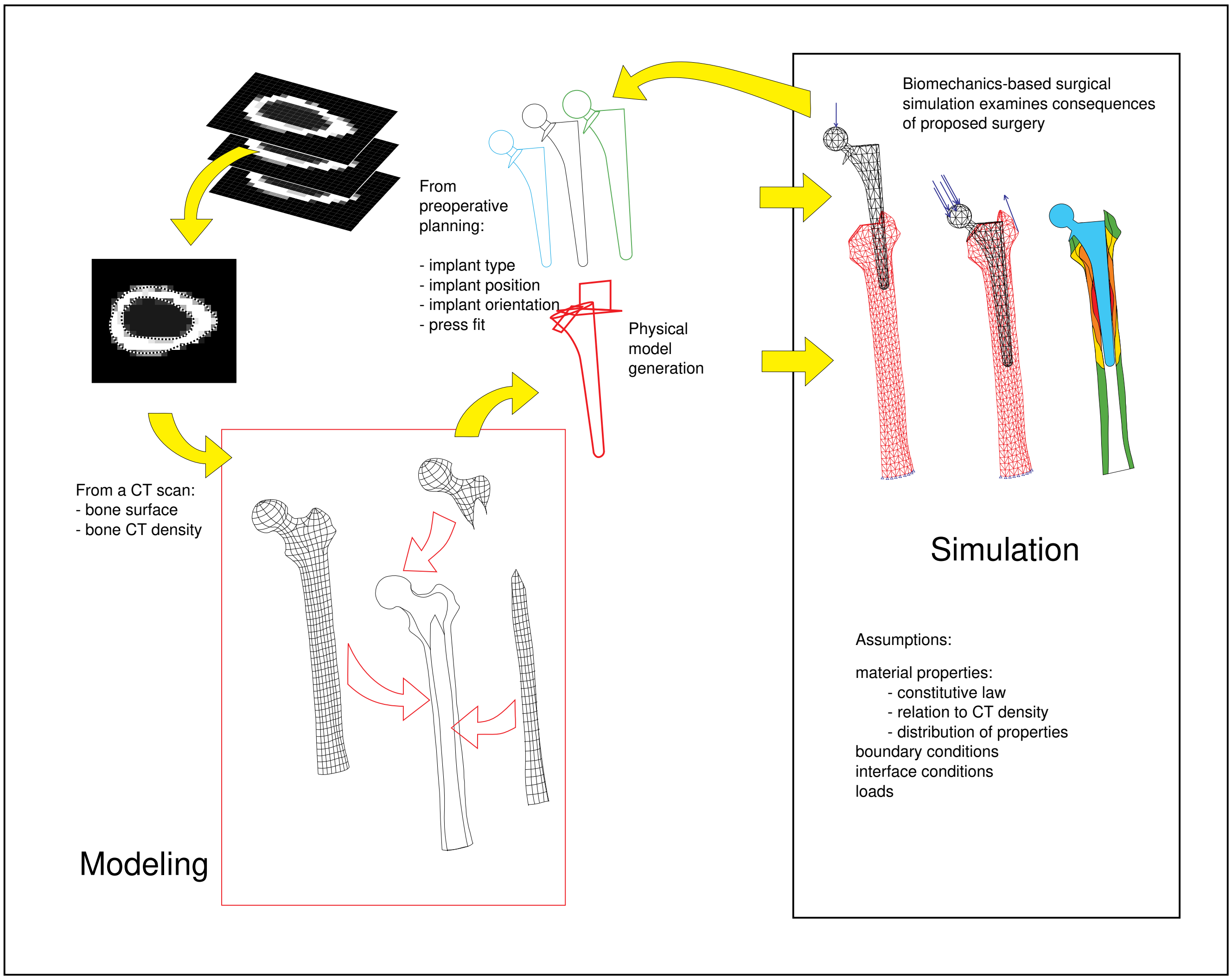


Planning - Biomechanical Simulation

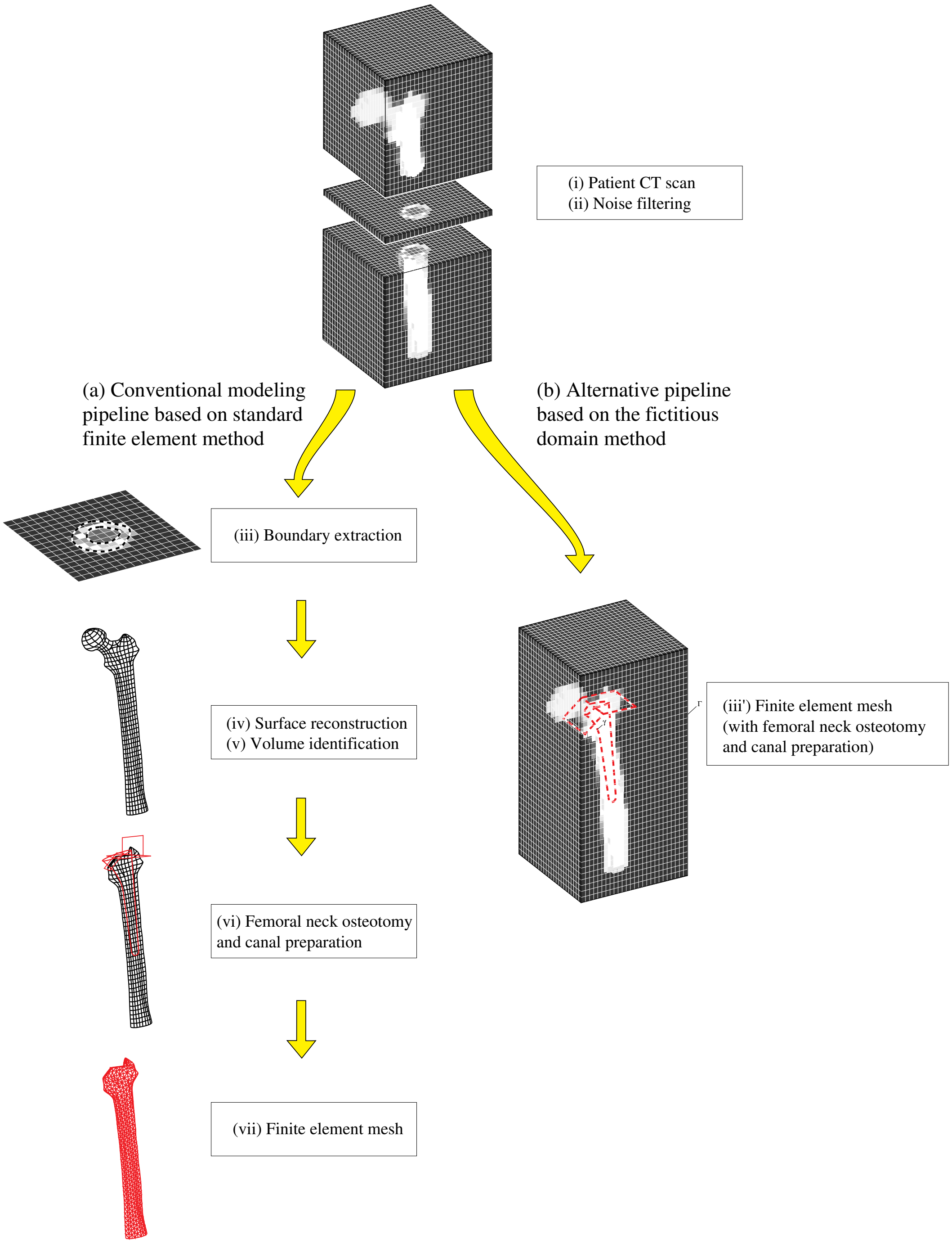


Execution - Real-Time Sensing and Guidance

Biomechanical Simulation

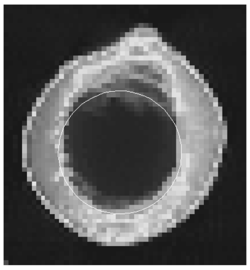


Structured vs Unstructured Modeling –3D

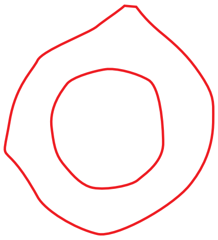


Structured vs Unstructured Modeling –2D

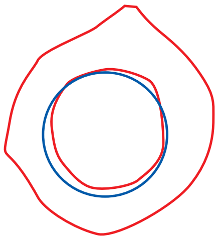
CT Scan

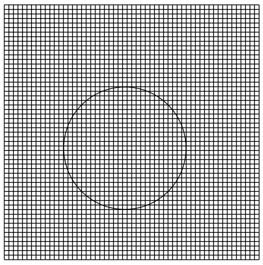


Boundary
Extraction



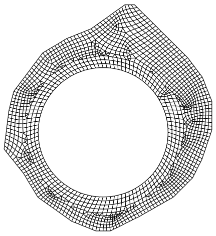
Canal
Preparation



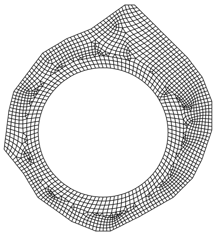


Physical
Model

Meshing

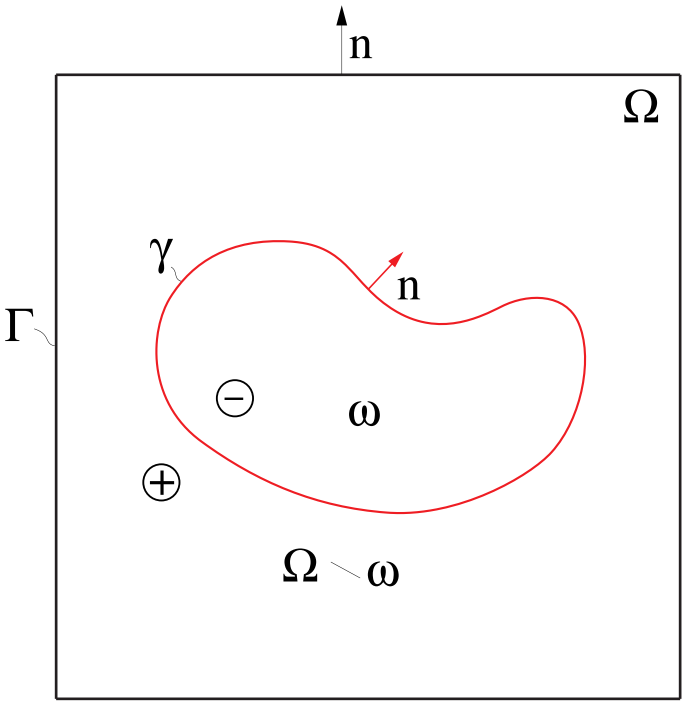


Querry material
database



Physical
Model

The Fictitious Domain Method –Concepts



$$\text{Given: } \left\{ \begin{array}{l} f \text{ in } \omega \\ g \text{ on } \gamma \end{array} \right\}$$

Problem P Problem $\hat{P}$

$$\text{Find } u : \left\{ \begin{array}{lll} \nabla^2 u = -f & \text{in} & \omega \\ u = g & \text{on} & \gamma \\ u_n = 0 & \text{on} & \Gamma \end{array} \right\} \xrightarrow[\text{entire domain}]{\text{extend over}} \text{Find } \hat{u} : \left\{ \begin{array}{lll} \nabla^2 \hat{u} = -\hat{f} & \text{in} & \Omega \\ \hat{u} = g & \text{on} & \gamma \\ \hat{u}_n = 0 & \text{on} & \Gamma \end{array} \right\}$$

with $\hat{f}|_\omega = f$

Then

$$\hat{u}|_\omega = u$$

The Fictitious Domain Method –Formulation

Variational Formulation of $\hat{P}$

$$\int_\Omega \nabla v \cdot \nabla \hat{u}\, d\Omega + \int_\gamma v\, \lambda\, d\gamma = \int_\Omega \hat{f} v\, d\Omega$$

$$\int_\gamma \mu(\hat{u} - g)\, d\gamma = 0$$

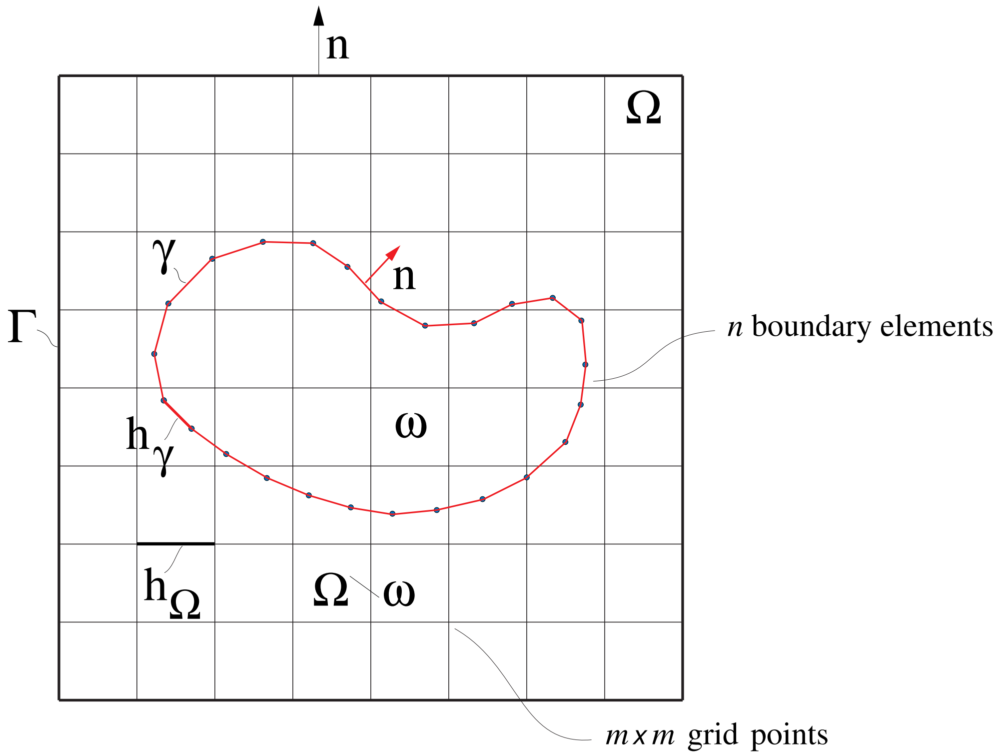
(enforcing weakly the
Dirichlet condition)

where

Lagrange multiplier: $\lambda = [\hat{u}_n]_\gamma \quad (= \hat{u}_n^+ - \hat{u}_n^-)|_\gamma$

References:
Fix G. M., *SIAM Rev.*, **18**(3), 460–484, 1976.
Astrakhantsev G. P., *USSR Comp. Math. and Math. Phys.*, **18**, 114–121, 1978.
Finogenov S.A. and Kuznetsov Y. A., *Sov. J. Numer. Anal. Math. Modeling*, **3**, 301–323, 1988.
Glowinski R., Pan T.-W., Wells, R.O. Jr., and Zhou X., *J. Comp. Phys.*, **126**, 40–51, 1996.
Glowinski R., Pan T.-W., and Periaux J., *Comput. Methods Appl. Mech. Engrg.*, **111**, 283–303, 1994.
Glowinski R., Pan T.-W., and Periaux J., *Comput. Methods Appl. Mech. Engrg.*, **112**, 133–148, 1994.

The Fictitious Domain Method –Discretization



- Variational Form:

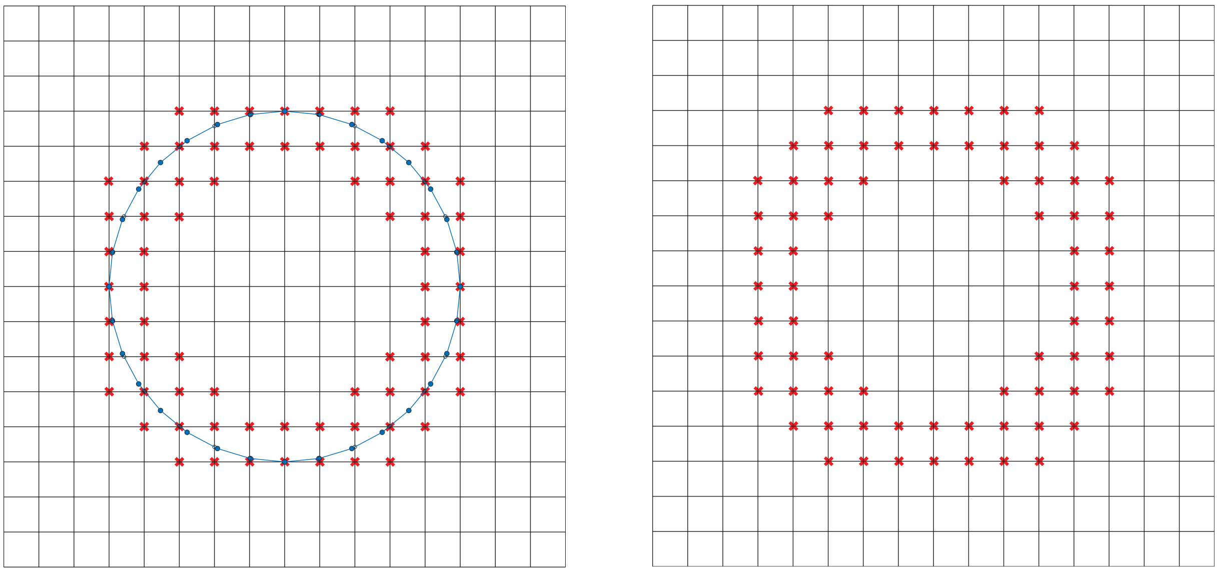
$$\int_{\Omega} \nabla v \cdot \nabla \hat{u} \, d\Omega + \int_{\gamma} v \, \lambda \, d\gamma = \int_{\Omega} \hat{f} v \, d\Omega$$

$$\int_{\gamma} \mu(\hat{u} - g) \, d\gamma = 0$$

- Introduce standard finite element approximations
- Respect LBB condition (eg, linear-constant pair of approximants)

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$$

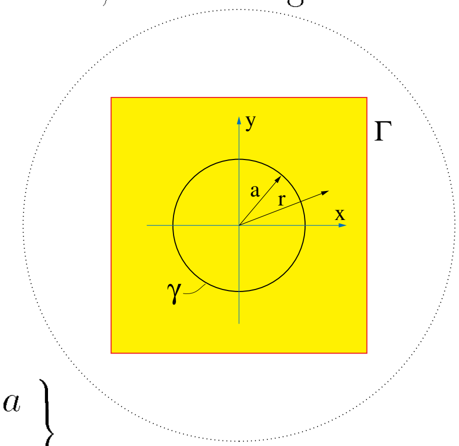
Role of B



Is the solution sensitive to
the choices for h_Ω, h_γ or h_Ω/h_γ?

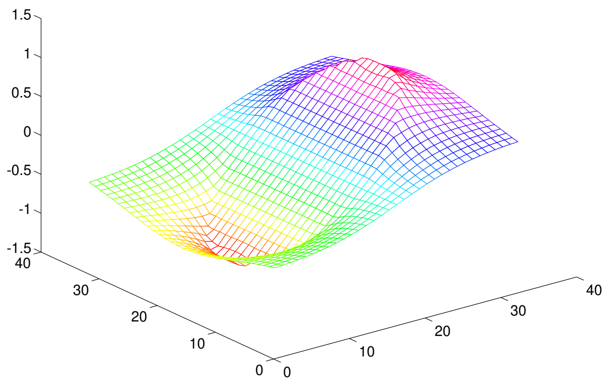
The Fictitious Domain Method –Example

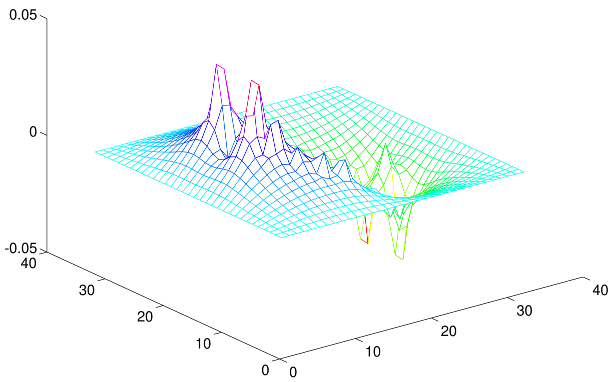
$$\nabla^2 u = 0, \text{ everywhere,}$$
$$u = u_0 \cos n\theta, \text{ on } r = a,$$
$$u = u_0 \frac{a^n}{a^{2n}+b^{2n}}[(x^2+y^2)^{\frac{n}{2}} + \frac{b^{2n}}{(x^2+y^2)^{\frac{n}{2}}}]\cos n\theta, \text{ on the edges}$$



Then

$$u = \begin{cases} u_0 \frac{r^n}{a^n}\cos n\theta & 0 \le r \le a \\ u_0 \frac{a^n}{a^{2n}+b^{2n}}\left[r^n + \frac{b^{2n}}{r^n}\right]\cos n\theta & r \ge a \end{cases}$$
$$\lambda|_a = \frac{u_0}{a}\frac{2b^{2n}}{a^{2n}+b^{2n}} n \cos n\theta$$





Is the solution/convergence sensitive to the values of λ?

The Fictitious Domain Method –Errors

Summary of L_2 errors ($e_0 = ||\hat{u} - \hat{u}_h||_0$)

- $\rho = \frac{h_\Omega}{h_\gamma} < \frac{1}{2}$ to avoid ill-conditioning (fine background grid)
- $\frac{h_\Omega}{h_\gamma} \longrightarrow 0$ when refining

For the linear-constant pair:

$\lambda = 0$	$O(h_\Omega^2)$
$\lambda \neq 0$	$O(h_\Omega)$

Application to Linear Elasticity –Formulation

$$\text{Find } \boldsymbol{u} : \left\{ \begin{array}{lll} \operatorname{div}\boldsymbol{S} + \boldsymbol{f} = \mathbf{o} & \text{in} & \omega \\ \boldsymbol{u} = \boldsymbol{g} & \text{on} & \gamma \\ \boldsymbol{S}\boldsymbol{n} = \boldsymbol{t} = \mathbf{o} & \text{on} & \Gamma \end{array} \right\} \text{ with } \begin{array}{l} \boldsymbol{S} = 2\mu_L\, \boldsymbol{E} + \lambda_L\, \boldsymbol{I}\, \operatorname{tr}\boldsymbol{E} \\ \boldsymbol{E} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) \end{array}$$

or

$$\text{Find } \hat{\boldsymbol{u}} : \left\{ \begin{array}{l} \int_\Omega \hat{\boldsymbol{S}} \cdot \nabla\boldsymbol{v} + \int_\gamma \hat{\boldsymbol{\lambda}} \cdot \boldsymbol{v} = \int_\Omega \hat{\boldsymbol{f}} \cdot \boldsymbol{v} + \int_\Gamma \hat{\boldsymbol{t}} \cdot \boldsymbol{v} \\ \int_\gamma \boldsymbol{\mu} \cdot \hat{\boldsymbol{u}} = \int_\gamma \boldsymbol{\mu} \cdot \boldsymbol{g} \end{array} \right\}$$

with

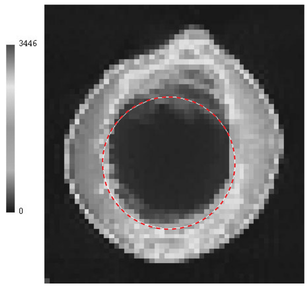
$$\begin{gathered} \hat{\boldsymbol{t}} = \hat{\boldsymbol{S}}\boldsymbol{n} \text{ on } \Gamma, \gamma \\ \hat{\boldsymbol{S}} = 2\mu_L\, \hat{\boldsymbol{E}} + \lambda_L\, \boldsymbol{I}\, \operatorname{tr}\hat{\boldsymbol{E}} \\ \hat{\boldsymbol{E}} = \frac{1}{2}\left(\nabla\hat{\boldsymbol{u}} + \nabla\hat{\boldsymbol{u}}^T\right) \\ \hat{\boldsymbol{f}}|_\omega = \boldsymbol{f} \end{gathered}$$

Then

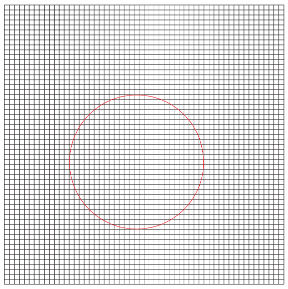
$$\hat{\boldsymbol{u}}|_\omega = \boldsymbol{u}$$

Application to Linear Elasticity –2D

2D Femoral CT-scan data

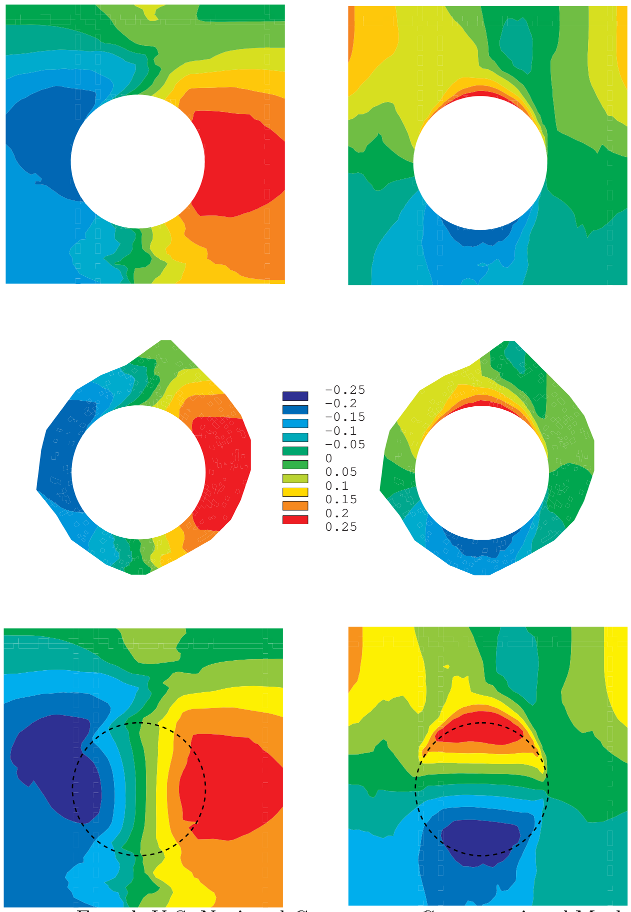


Femoral CT-scan slice



Problem geometry

x-displacements y-displacements



Pseudo-structured

Unstructured

Structured

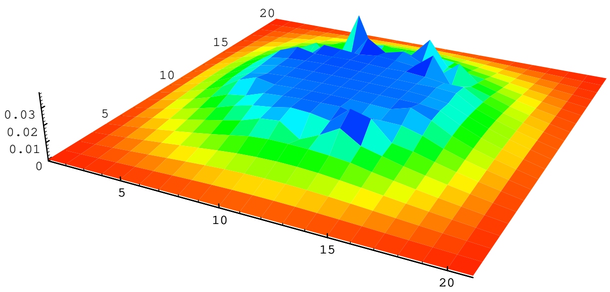
Three-Dimensional Test Cases (1)

Dirichlet conditions on spherical boundaries for:

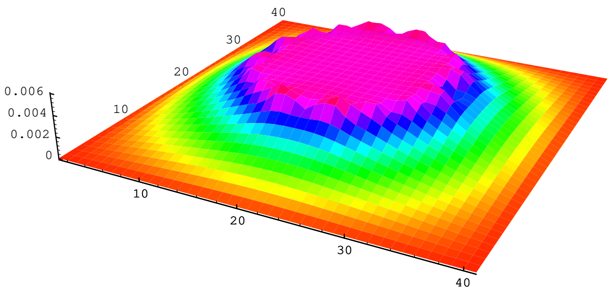
- Poisson's equation

Zero Lagrange Multiplier

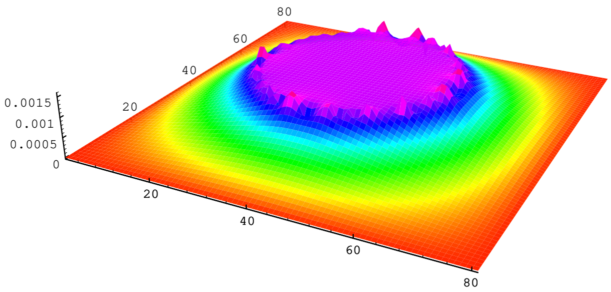
Error on plane z=2.



Error on plane z=2.

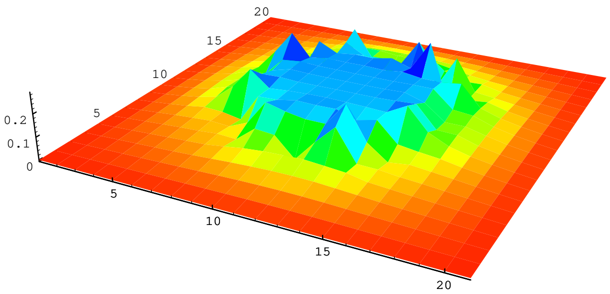


Error on plane z=2.

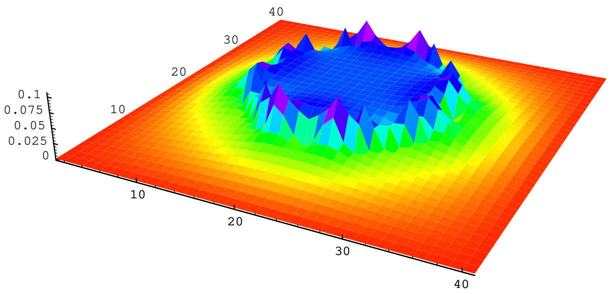


Non-zero Lagrange multiplier

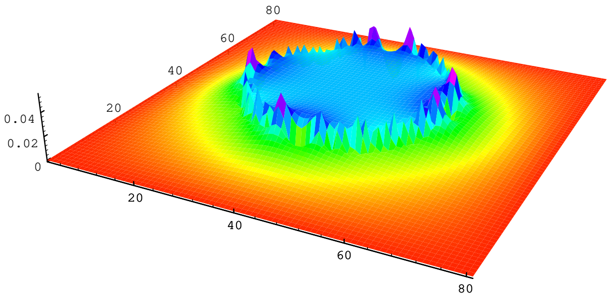
Error on plane z=2.



Error on plane z=2.



Error on plane z=2.

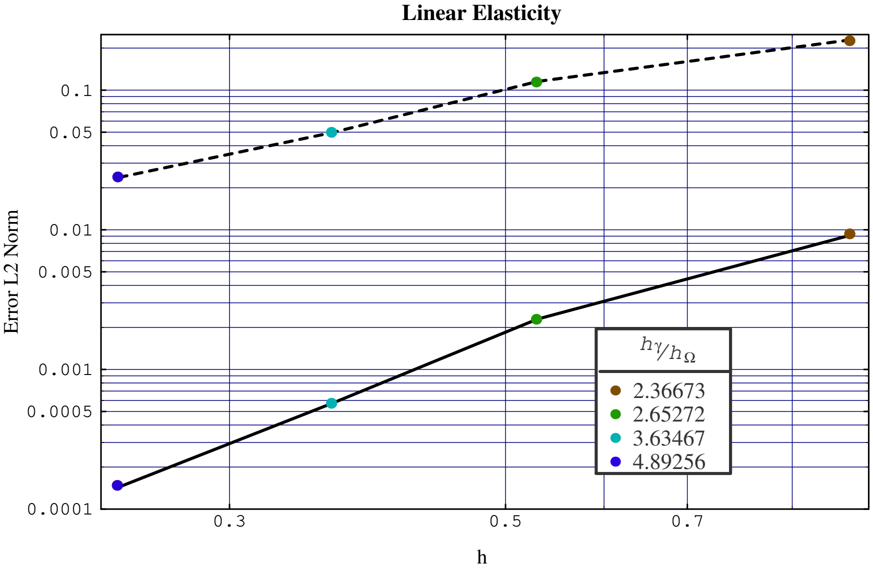


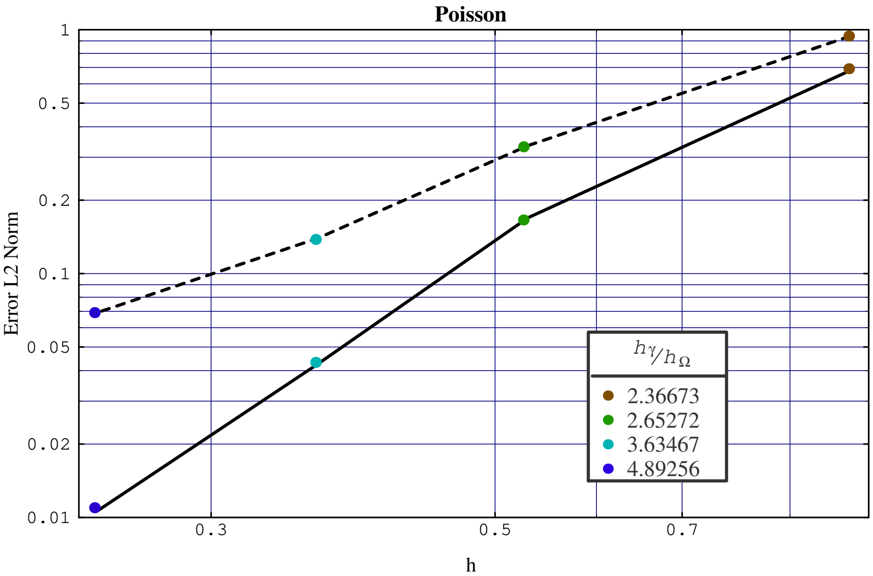
Three-Dimensional Test Cases (2)

Dirichlet conditions on spherical boundaries for:

- Poisson's equation, and
- Linear elasticity

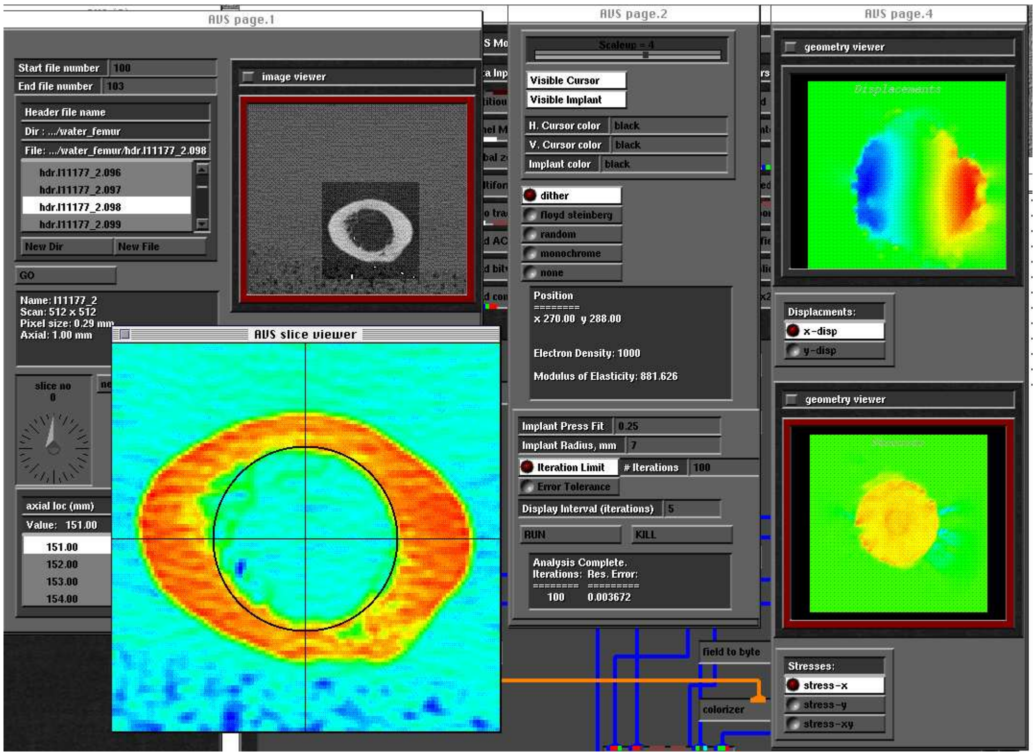
Convergence
Solid line: Lagrange multiplier $= 0$
Dashed line: Lagrange multiplier $\neq 0$





Pre-Operative THR Planner –2D

- Enhanced with biomechanical simulation module
- 30,000 unknowns
- Near real-time performance on a high-end workstation



Conclusions

- Advantages
  - Structured grids –Allows for fast solvers, reduced storage
  - Shorter modeling cycle
  - Bypassing of geometry-reconstruction errors
  - Easy automation for patient-specific modeling
  - Natural choice for grid-based data
- Disadvantages
  - Sub-optimal convergence rate
  - Design of preconditioners